

Non-universal parameters, corrections and universality in Kardar-Parisi-Zhang growth

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Abstract. We present a comprehensive numerical investigation of non-universal parameters and corrections related to interface fluctuations of models belonging to the Kardar-Parisi-Zhang (KPZ) universality class, in $d = 1 + 1$, for both flat and curved geometries. We analyzed two classes of models. In the isotropic models the non-universal parameters are uniform along the surface, whereas in the anisotropic growth they vary. In the latter case, that produces curved surfaces, the statistics must be computed independently along fixed directions. The ansatz $h = v_\infty t + (\Gamma t)^{1/3} \chi + \eta$, where χ is a Tracy-Widom (geometry-dependent) distribution and η is a time-independent correction, is probed. Our numerical analysis shows that the non-universal parameter Γ determined through the first cumulant leads to a very good accordance with the extended KPZ ansatz for all investigated models in contrast with the estimates of Γ obtained from higher order cumulants that indicate a violation of the generalized ansatz for some of the studied models. We associate the discrepancies to corrections of unknown nature, which hampers an accurate estimation of Γ at finite times. The discrepancies in Γ via different approaches are relatively small but sufficient to modify the scaling law $t^{-1/3}$ that characterize the finite-time corrections due to η . Among the investigated models, we have revisited an off-lattice Eden model that supposedly disobeyed the shift in the mean scaling as $t^{-1/3}$ and showed that there is a crossover to the expected regime. We have found model-dependent (non-universal) corrections for cumulants of order $n \geq 2$. All investigated models are consistent with a further term of order $t^{-1/3}$ in the KPZ ansatz.

1. Introduction

Dynamics of self-affine surfaces is a fascinating topic of non-equilibrium statistical physics where pattern formation, stochasticity, scale invariance and universality are joined in a unified framework [1, 2]. The KPZ equation [3]

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \xi, \quad (1)$$

proposed by Kardar, Parisi and Zhang in 1986, emerged as one of the most prominent theoretical systems and established a universality class which encompasses a plenty of models [1, 2]. This equation describes the non-conservative evolution of an interface $h(x, t)$ subjected to a white noise ξ defined by $\langle \xi \rangle = 0$ and $\langle \xi(x, t) \xi(x', t') \rangle = D \delta(x - x') \delta(t - t')$. The KPZ universality class goes beyond the surface dynamics including

theoretical studies in random polymers [4] and fluid transport [5]. The understanding of the KPZ universality class has undergone a noticeable progress during the last few years with its experimental realization [6, 7, 8, 9] as well as the achievement of analytical solutions [4, 10, 11, 12, 13, 14] of the KPZ equation in 1+1 dimensions.

Self-affine interface evolution is featured by a dynamical regime where the interface width w , defined as the standard deviation of the surface heights, $w \equiv \sqrt{\langle h^2 \rangle - \langle h \rangle^2}$, increases in time as a power law $w \sim t^\beta$, and a stationary regime where the interface width depends on the system size as $w \sim L^\alpha$. β and α are the growth and roughness exponents [1], respectively, that assume the exact values $\beta = 1/3$ and $\alpha = 1/2$ for KPZ class in $d = 1 + 1$ [3]. The universality in KPZ systems includes other important universal quantities related to the height distributions (HDs) [7, 15]. Motivated by analytical determination of the HDs of some models of the KPZ class, namely, the asymmetric exclusion process [16] and polynuclear growth model [17, 18], Prähofer and Spohn [17] conjectured that the heights for any KPZ system are asymptotically given by

$$h = v_\infty t + s_\lambda (\Gamma t)^{1/3} \chi \quad (2)$$

where $s_\lambda = \text{sgn}(\lambda)$, v_∞ and Γ are non-universal parameters while χ is a stochastic variable with universal properties. Moreover, the conjecture states that, in the dynamical regime, χ depends on the growth geometry and is given by Tracy-Widom (TW) distributions from random matrix theory [19] for flat and curved growth geometries. The Gaussian orthogonal ensemble (GOE) is expected for the former while Gaussian unitary ensemble (GUE) is expected for the latter. Therefore, the KPZ universality class splits into subclasses depending on the growth geometry. This conjecture has been verified by many analytical results [4, 10, 11, 13, 14], computer simulations [20, 21, 22, 23] and, most excitingly, in experiments [6, 7, 8, 9].

The TW distributions are conjectured for the asymptotic regime, but many experimental [6, 8], analytical [10, 24] and numerical [21, 22] works have shown the existence of a slow convergence with apparently universal properties. More specifically, the rescaled distribution $P(q)$, where

$$q = \frac{h - v_\infty t}{s_\lambda (\Gamma t)^{1/3}}, \quad (3)$$

has a shift in relation to the TW distributions that vanishes as

$$\langle q \rangle - \langle \chi \rangle \sim t^{-1/3}. \quad (4)$$

An exception to the correction $t^{-1/3}$ was obtained in the partially asymmetric simple exclusion process (PASEP) for the asymmetry parameter $p_c = 0.7822787862 \dots$ [24]. The PASEP result suggests that corrections different from $t^{-1/3}$ would appear in very particular situations. However, Takeuchi [23] reported off-lattice simulations of a radial Eden model where the first cumulant of the scaled distributions converges to the asymptotic GUE value as $t^{-2/3}$, in contrast with other Eden versions [21]. It is somehow intriguing that a stochastic model without control parameters does not have

the usual correction. Still in reference [24], a deterministic shift η was found in the polynuclear growth model (PNG) and in the totally/partially asymmetric exclusion processes (TASEP/PASEP), generalizing equation (2) to

$$h = v_\infty t + s_\lambda (\Gamma t)^{1/3} \chi + \eta + \dots \quad (5)$$

The correction $t^{-1/3}$ in the first cumulant derives directly from equation (5). It was also shown that higher order moments have corrections $\mathcal{O}(t^{-2/3})$ [24] and, consequently, all higher order cumulants of order $n \geq 2$ decays as $\mathcal{O}(t^{-2/3})$ or faster. Sasamoto and Spohn found out a result similar to equation (5) in their solution of the KPZ equation [10, 12] with a difference that η is random, but still independent of χ .

Many models belonging to the KPZ class are analytically intractable with our current knowledge and Monte Carlo simulations become the best accessible method to probe the universality of their interface fluctuations. The analysis of flat growth can be efficiently performed using lattice models [22, 25] whose computer implementations are commonly simple, relatively quick and low memory demanding. Eden model [26] and its variations [2] are basic examples of radial growth belonging to the KPZ class. However, it is well known that radial growth in lattices is distorted by anisotropy effects [27, 28, 29, 30]. Therefore, off-lattice Eden simulations [21, 23, 31] or suitable tricks using tuning parameters [21, 30] are needed to investigate interface fluctuations in the entire surface. Although off-lattice simulations of Eden growth are quite affordable in 1+1 dimensions [32], the generalization to higher dimensions is cumbersome [33].

In this work, we present extensive simulations of models belonging to the KPZ universality class to address the validity of the extended KPZ conjecture including the correction η given in equation (5). We investigated models exhibiting either isotropic or anisotropic growth. In the isotropic growth the velocity is the same for all parts of the interface such that the entire profile can be used to perform statistics. On other hand, in an anisotropic growth, in which the interface velocity varies along the interface, we must analyze fixed directions independently and, consequently, a large ensemble of samples are required since one or a few points from each sample are used for statistics.

We have found that the procedure to determine the model-dependent parameter Γ via second cumulant may be troublesome due to strong and/or puzzling corrections to the scaling. We adopted an alternative method using the first cumulant derivative, that exhibits a monotonic convergence to the asymptotic value with a correction $t^{-2/3}$ in all investigated models. This method allowed to determine shifts in the first cumulants always consistent with a decay $t^{-1/3}$, reinforcing the generality of the KPZ ansatz. We also investigated the convergence of higher order cumulants and identified complex and non-universal behaviors. Finally, an additional term $t^{-1/3}$ in the KPZ ansatz was found in all models considered here.

The paper is organized as follows. Section 2 presents the analysis of flat growth models concomitantly with the description of the numerical recipes applied in the present work. Sections 3 and 4 are devoted to isotropic and anisotropic radial growth, respectively. Section 5 presents the analysis of the droplet growth. We draw our

concluding remarks in section 6.

2. Isotropic flat growth

We consider two versions of the restricted solid-on-solid (RSOS) model [34] and the ballistic deposition (BD) model [1] on initially flat substrates. Averages were computed using 10^4 samples and system sizes $L = 2^{18}$ resulting in more than 2×10^9 data points for the statistics.

The RSOS model is defined as a random deposition obeying the height restriction $\Delta h = |h_j - h_{j+1}| \leq m$. In the BD model, particles move normally to the substrate and permanently attach to the first nearest neighbor contact with a previously deposited particle. In both models, periodic boundary conditions are assumed and the time is increased by $\Delta t = 1/L$ for each deposition attempt.

Accurate estimates of the non-universal parameters are imperative to a reliable characterization of the corrections. The asymptotic velocity is accurately determined taking the time derivative of the mean height in equation (5):

$$\langle h \rangle_t = v_\infty + \frac{s_\lambda \Gamma^{1/3} \langle \chi \rangle}{3} t^{-2/3}. \quad (6)$$

Then, the velocity v_∞ can be extracted by extrapolating $\langle h \rangle_t$ versus $t^{-2/3}$ in a linear regression for $t \rightarrow \infty$. The inset of figure 1 shows a typical plot used to determine v_∞ for RSOS model with $m = 1$. Interface velocities for all investigated isotropic models are shown in table 1.

The parameter Γ , in terms of the constants of the KPZ equation, is given by $\Gamma = \frac{1}{2} A^2 |\lambda|$ with $A = D/\nu$ [15, 8]. The parameters λ and A can be determined

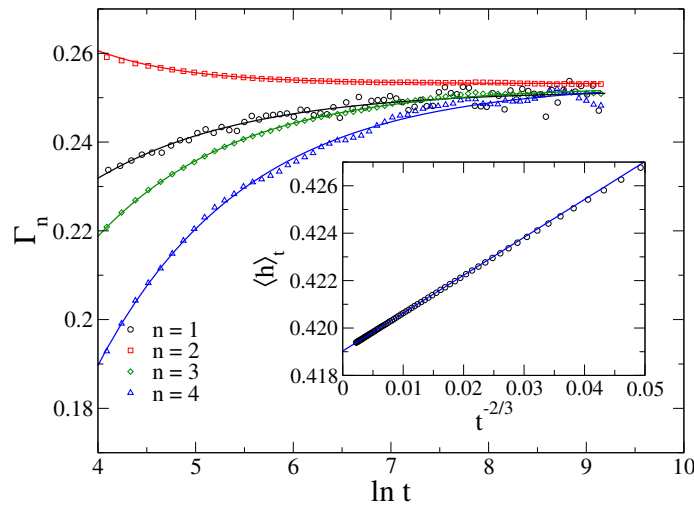


Figure 1. Numerical determination of the parameter Γ for the RSOS model grown on flat substrates, with $\Delta h \leq 1$, using different cumulants. Solid lines are non-linear regressions. The inset shows the interface velocity against $t^{-2/3}$ (symbols) and the linear regression used to determine the asymptotic velocity.

model	s_λ	v_∞	Γ_1	Γ_2	Γ_3	Γ_4	$\langle\eta\rangle$
RSOS (m=1)	-1	0.419030(3)	0.252(1)	0.2532(3)	0.2525(2)	0.2534(3)	-0.32(4)
RSOS (m=2)	-1	0.60355(1)	0.812(2)	0.816(9)	0.811(2)	0.815(2)	-0.66(6)
BD	1	2.13983(1)	4.94(1)	4.778(2)	4.799(7)	4.74(2)	-0.9(1)
Eden D	1	0.51371(2)	1.00(1)	≈ 1	0.99(1)	0.98(2)	0.50(5)

Table 1. Non-universal parameters for distinct isotropic growth models. The number in parenthesis represents the uncertainties obtained from the non-linear regressions. The parameter Γ_2 does not have uncertainty for off-lattice Eden model due to the lack of a monotonic convergence.

numerically using the tilt dependence of the growth velocity and the amplitude of a two-point correlation function, respectively [35]. Alternatively, Γ can be obtained directly from equation (2) assuming that the cumulants of χ are those of TW distributions. The scaled second cumulant

$$g_2 = \frac{\langle h^2 \rangle_c}{t^{2/3}} \rightarrow \Gamma^{2/3} \langle \chi^2 \rangle_c, \quad (7)$$

where $\langle X^n \rangle_c$ denotes the n th cumulant of X , has been used to this purpose [6, 23, 7, 22]. In principle, one can obtain Γ from any higher order cumulant since equation (2) yields

$$g_n = \frac{\langle h^n \rangle_c}{s_\lambda^n t^{n/3}} \rightarrow \Gamma^{n/3} \langle \chi^n \rangle_c, \quad n \geq 2 \quad (8)$$

Alternatively, the parameter Γ can also be obtained from the first cumulant since

$$g_1 = s_\lambda 3(\langle h \rangle_t - v_\infty) t^{2/3} \rightarrow \Gamma^{1/3} \langle \chi \rangle. \quad (9)$$

Although widely used in many experimental [8, 6, 7] and computer [22, 23] studies, the determination of Γ using second order cumulants has complications when there are statistical dependencies among χ , η or other unknown terms in equation (5). In this case, crossed terms appear in cumulants leading to relevant corrections that, in principle, may be puzzling. Otherwise, the analysis using $\langle h \rangle_t$ is free from crossed terms and does not depend on η . Noise in numerical derivatives counts against this last method. We propose that the first cumulant derivative yields more reliable estimates of Γ than $\langle h^2 \rangle_c$ does, as discussed in the this and in the following sections.

We denote by $\Gamma_n = (g_n / \langle \chi^n \rangle_c)^{3/n}$, the value of Γ estimated via the n th cumulant. Figure 1 shows the curves used to determine the non-universal parameters of the RSOS model with $m = 1$. Independently of the used cumulant, the data have finite-time corrections. To estimate the asymptotic value as well as the scaling of the correction, numerical values of $\Gamma_n(t)$, that are recorded exponentially spaced in time, are plotted as function of $s = \ln t$. Assuming that there is a power law correction, we performed non-linear regressions in the form

$$\Gamma_n = \Gamma_n(\infty) + c \exp(-a_n s), \quad (10)$$

discarding very short times. The extrapolated values for Γ_n are shown in table 1. All cumulants yield essentially the same asymptotic value for Γ_n , that is in very good

agreement with our previous estimate using longer growth times (and less samples) without extrapolations [22]. The correction in Γ_1 has an exponent $a_1 = 0.661 \approx 2/3$. Assuming that the next term in equation (5) is a power law $\zeta t^{-\gamma}$, we have that

$$g_1 = \Gamma^{1/3} \langle \chi \rangle + \langle \zeta \rangle t^{-\gamma-1/3}. \quad (11)$$

So, a correction $a_1 = 2/3$ in g_1 , or equivalently in Γ_1 , shows that the next leading term in equation (5) decays as $t^{-1/3}$. The correction in Γ_2 , that have the same exponent as $\langle q^2 \rangle_c - \langle \chi^2 \rangle_c$, is $a_2 \approx 1$ while $a_3 \approx a_4 \approx 2/3$ are found for the third and forth order cumulants. These results are corroborated in figure 2, where the differences between g_n , $n = 1, \dots, 4$, and their asymptotic values (see table 2) are plotted against time. It is worth mentioning that corrections in cumulants were already studied for RSOS model [22] and are consistent with the present results.

The power law correction $t^{-1/3}$ in the mean is also verified with a good precision as shown in the inset of figure 2, where the difference between the scaled height and the GOE first cumulant has a plateau when rescaled by the expected power law. Considering the variation of the plateau inside errors of Γ and v_∞ , the estimated amplitude of the correction is $\langle \eta \rangle \approx 0.32(4)$. The mean $\langle \eta \rangle$ for all isotropic models are shown in table 1.

We also simulated the RSOS model allowing a height difference $\Delta h \leq 2$. The results are essentially the same except by a non-monotonic time-dependence for Γ_2 that initially decays and then increases towards the asymptotic value also with a correction faster than $t^{-2/3}$. For other g_n , the corrections are approximately $t^{-2/3}$, the same as in the $m = 1$ case. The non-universal parameters for $m = 2$ and cumulant ratios are also shown in tables 1 and 2.

Figure 3 shows the curves Γ_n against time and the respective non-linear regressions

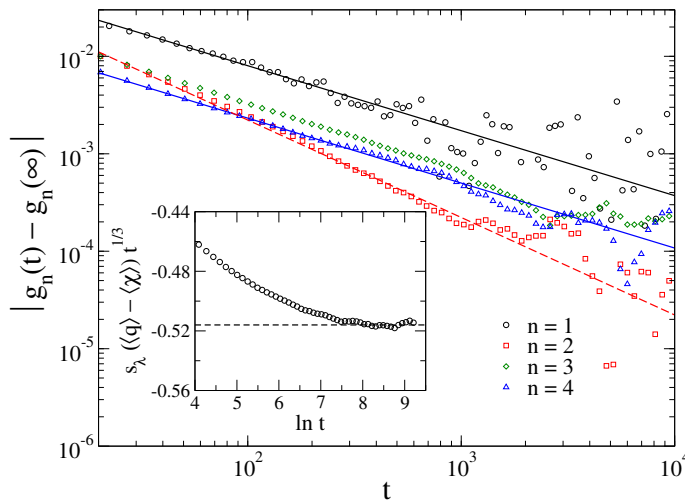


Figure 2. Corrections in g_n for the RSOS model grown on flat substrates. Dashed line represents a power law t^{-1} and the solid ones represent the power law $t^{-2/3}$. Inset shows the difference between the first cumulant of equation (3) and the GOE mean $\langle \chi \rangle = -0.76007$ using $v_\infty = 0.419030$ and $\Gamma = 0.252$. The shift is scaled by the theoretic power law $t^{-1/3}$.

used to obtain their asymptotic values for the BD model. Surprisingly, the extrapolation from Γ_1 is a little larger than the others. The difference of 3-4% is sufficient to affect the scaling law ruling the shift of the distribution in relation to GOE as shown in the inset of figure 3. The value $\Gamma = 4.94$ obtained from the first cumulant derivative yields an excellent agreement with a correction $t^{-1/3}$ while the values of Γ extrapolated from higher order cumulants do not.

The cause of this difference is unknown. A possibility could be crossed terms involving random variables in higher order cumulants, that are not present in the first one. These terms would introduce complicated corrections leading to a non-monotonic convergence and Γ could not be properly extrapolated using equation (10) for $n \geq 2$. Another possible explanation is that the asymptotic distribution of q for BD model does not converge to GOE but to a shifted GOE distribution with $\chi + a$ where a is a deterministic and non-universal parameter. In this case, the deterministic shift a cannot be obtained using higher order cumulants since $\langle a^n \rangle_c = 0$ for $n \geq 2$. If this last hypothesis is correct, we would have $\Gamma_1 = \Gamma(1 + a/\langle\chi\rangle)^3$ that, according to the parameters of table 1, provides a tiny shift $a \approx 0.01$. Such a small shift is easily unnoticed in scaled height distributions obtained in simulations or experiments. Indeed, the correction $t^{-1/3}$ shown in the inset of figure 3 is equally obtained in curves $\langle q \rangle - \langle \chi + a \rangle$ against t (data not shown). It is important to emphasize that this shift was not predicted in the other KPZ systems investigated either in the present work or elsewhere. Independently of the numerical evidence, we believe that the non-monotonic convergence is the most likely scenario.

In reference [22], we have showed that the shift in the first cumulant is consistent with the usual $t^{-1/3}$ decay using the parameter $\Gamma_2 = 4.90(5)$ obtained from $\langle h^2 \rangle_c$ in simulations up to $t = 5 \times 10^4$ without extrapolation. We checked that the extrapolation

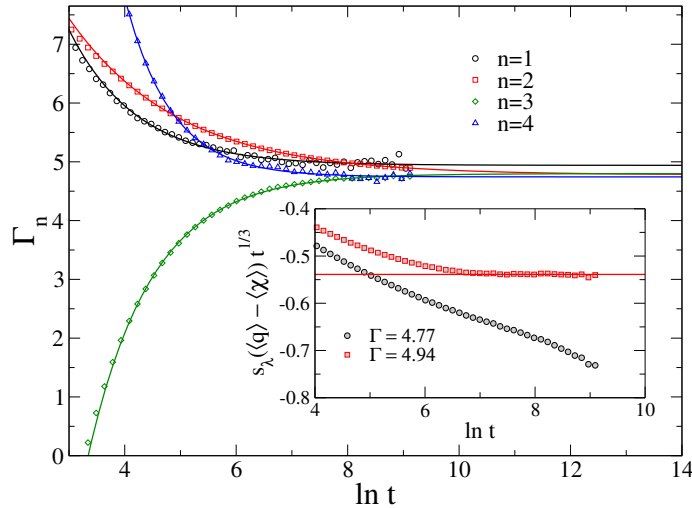


Figure 3. Determination of the non-universal parameter Γ for the BD model grown on flat substrates. Solid lines are non-linear regressions. Inset: Shift in the first cumulant scaled by the expected power law $t^{-1/3}$ using two estimates of Γ and $v_\infty = 2.13983$.

of the data of reference [22] is consistent with our current estimate of Γ_2 . A careful view in the double-logarithmic plot presented in reference [22] reveals a slightly bent curve indicating a correction not so close to $t^{-1/3}$ as wished. Also, the uncertainties in Γ and v_∞ reported there are much larger than the current ones such that the propagated error in the power law is sufficiently large to embrace the exponent $-1/3$.

Corrections in g_n for BD model are shown in figure 4. In g_1 , we have found an exponent $a_1 \approx 2/3$, the same value found for the RSOS model. Nevertheless, an unusual exponent $a_2 \approx 1/2$ was observed in second order cumulant. To our knowledge, there is not analytical or experimental analogues for this correction. This exponent is at odds with our previous analysis reporting a decay faster than $t^{-2/3}$ in plots $\langle q^2 \rangle_c - \langle \chi^2 \rangle_c$ against time [22]. The discrepancy is due to the over-estimation of Γ_2 used in the rescaled height q while in the current analysis, the dependence on Γ_2 is implicit to g_2 and, therefore, does not involve error propagation. The third and fourth order cumulants decay with exponents $a_3 \approx 0.9$ and $a_4 \approx 1.25$, respectively, that are consistent with the previous report $\mathcal{O}(t^{-2/3})$ or faster [22]. These results show that the exponents measured in the power laws related to the finite-time corrections are very sensitive to Γ and, thus, a precise characterization of the corrections demands accurate estimates of Γ .

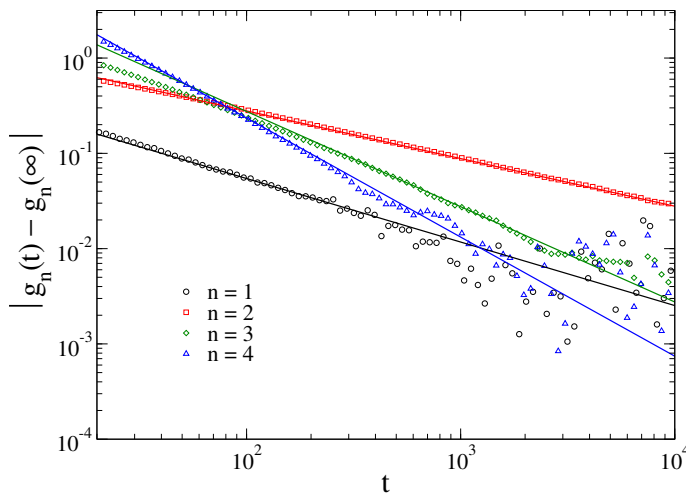


Figure 4. Corrections in g_n for the BD model grown on flat substrates. The lines are power laws $t^{-2/3}$ ($n = 1$), $t^{-1/2}$ ($n = 2$), t^{-1} ($n = 3$), and $t^{-1.25}$ ($n = 4$).

3. Isotropic radial growth

We simulated radial growth using an Eden model proposed in reference [23], that we have named as Eden D [33] since A, B, and C versions are already defined in literature [2, 30]. The model is defined as follows: In each time step, a particle of the cluster and a position tangent to it are randomly chosen. A new particle is added to the chosen position if this event does not imply the overlap with any other particle of the cluster. In the case of overlap, simulation proceeds to the next step. The time is increased by $\Delta t = 1/N$, where

N is the number of particles of the cluster at time t . Optimization strategies described in reference [32] were used to speed up the simulation. A cluster of diameter 8000 takes typically 8 min of simulation in a CPU Intel Xeon 3.2 GHz, whereas if no optimization is used the same simulation takes several hours of computation. We simulated the same version of the Eden model proposed in reference [23] using a much better statistics (90000 samples against 3000) and much longer growth times ($t \simeq 8000$ against 2000). Considering the same interval used in reference [23] ($250 < t < 2000$), we have found $v_\infty = 0.51390$ in full agreement with $v_\infty = 0.5139(2)$ reported there. However, a more accurate estimate $v_\infty = 0.51371(1)$ was found for $t > 2000$ (bottom inset of figure 5).

Curves used to determine $\Gamma_n(\infty)$ for the Eden model are shown in figure 6. The second order cumulant depends non-monotonically on time as highlighted in the inset of figure 6. This non-monotonicity hampers an extrapolation to $\Gamma_2(\infty)$ using equation (10). Using an extrapolation similar to that used in reference [23], *i.e.*, forcing a correction $a_2 = 2/3$ in Γ_2 , we obtained $\Gamma_2 \simeq 0.995$ considering the time interval $1000 < t < 8000$. This value is slightly smaller than $\Gamma_2 = 1.02(2)$ reported in reference [23]. The first cumulant derivative yields a monotonic convergence with an exponent $a_1 \approx 2/3$ and extrapolates to $\Gamma_1 = 1.00(1)$. Moreover, Γ_3 and Γ_4 also have corrections consistent with $t^{-2/3}$ and converge to values very close to 1 (see table 1).

The main plot of figure 5 shows the shift $\langle q \rangle - \langle \chi \rangle$ against time for the Eden model. Using the velocity $v_\infty = 0.51390$ obtained by Takeuchi, his result, a decay much faster than $t^{-1/3}$ [23], is reproduced. If the more accurate estimate $v_\infty = 0.51371$ is used instead, a decay close to $t^{-2/3}$ is observed for short times and is followed by a crossover to a smaller exponent. The top inset of figure 5 shows the local exponent against time. One can clearly see that the scaling exponent has not reached a stationary value even

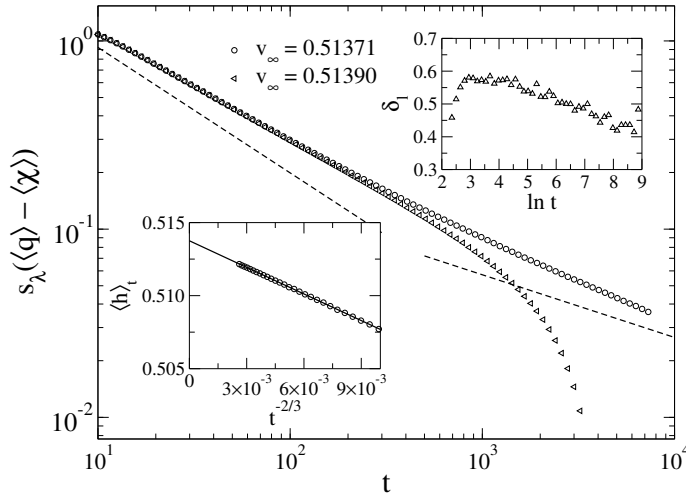


Figure 5. Correction in the first cumulant of the scaled height in relation to the corresponding GUE value $\langle \chi \rangle = -1.77109$ for off-lattice Eden D model. The parameters used were $\Gamma = 1.00$ and interface velocities indicated in the legends. Dashed lines are power laws $t^{-2/3}$ and $t^{-1/3}$ as guides to the eyes. Top inset shows the local exponent against time, while the bottom one shows the velocity against $t^{-2/3}$.

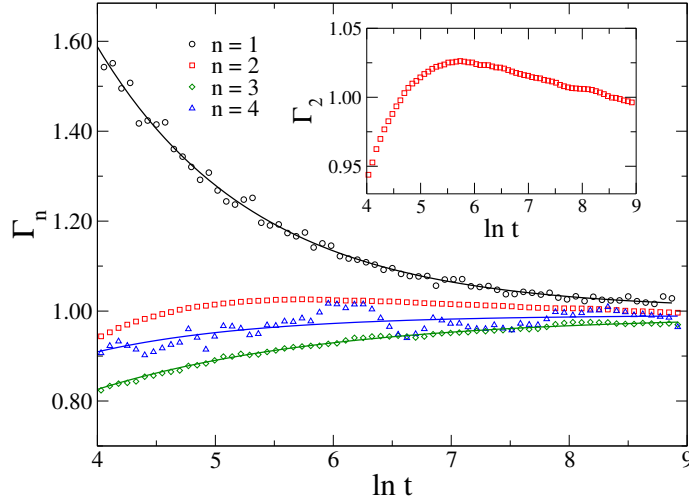


Figure 6. Determination of the non-universal parameter Γ for the Eden model using different cumulants. Solid lines are non-linear regressions used to extrapolate Γ_n . Inset: Zoom to emphasize the non-monotonicity of Γ_2 .

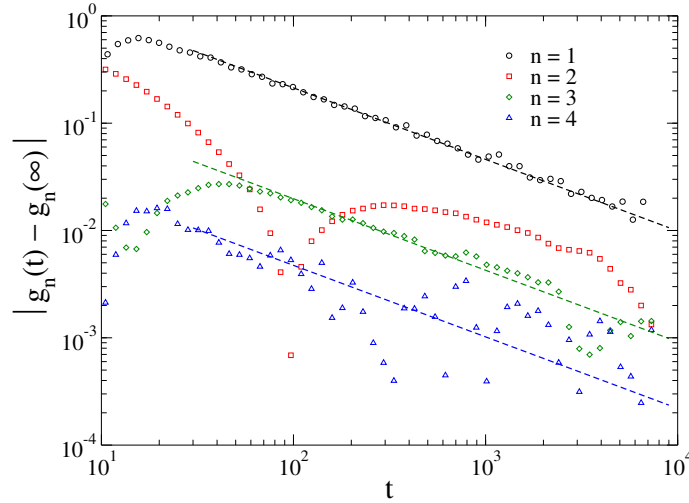


Figure 7. Corrections in scaled cumulants for Eden model. Dashed lines represent the power law $t^{-2/3}$ as guides to eyes.

for our longest simulated times.

Corrections in g_n are shown in figure 7. The quantity g_1 has an excellent agreement with a $t^{-2/3}$ power law resulting again in a correction $t^{-1/3}$ in the KPZ ansatz given by equation (5). Thus, a higher order term $t^{-2/3}$ is expected in $\langle q \rangle - \langle \chi \rangle$ and explains the short-time decay observed in figure 5. Performing a double power law regression $\langle q \rangle - \langle \chi \rangle = at^{-1/3} + bt^{-2/3}$, an excellent fit is obtained with positive amplitudes $a = 0.50(5)$ and $b = 4.0(1)$. This result shows that the amplitude of the correction η is much smaller than the next leading term resulting a very slow crossover to the asymptotic scaling law $t^{-1/3}$. Due to the lack of monotonicity, scaling of the corrections in g_2 could not be determined with the present data. The correction in the third cumulant is $t^{-2/3}$. Despite of large fluctuation, the forth cumulant also decays

consistently with $t^{-2/3}$.

Even though all parameters in table 1 depend on the non-universal parameter Γ , they can be combined in dimensionless cumulant ratios that must reflect the universality of χ . The relative variance R , the skewness S and the kurtosis K defined by

$$R = \frac{g_2}{g_1^2} = \frac{\langle \chi^2 \rangle_c}{\langle \chi \rangle^2}, \quad (12)$$

$$S = \frac{g_3}{g_2^{3/2}} = \frac{\langle \chi^3 \rangle_c}{\langle \chi^2 \rangle_c^{3/2}}, \quad (13)$$

and

$$K = \frac{g_4}{g_2^2} = \frac{\langle \chi^4 \rangle_c}{\langle \chi^2 \rangle_c^2}, \quad (14)$$

must therefore be model-independent.

Table 2 shows the cumulant ratios for all isotropic models investigated. The results are in excellent agreement with the corresponding GOE ($R_{goe} = 1.1046$, $S_{goe} = 0.2935$, and $K_{goe} = 0.1652$) for flat models and GUE ($R_{gue} = 0.2592$, $S_{gue} = 0.2241$, and $K_{gue} = 0.09345$) for the radial Eden model. Scaled height distributions $P(q)$ for all isotropic growth models were reported elsewhere [22, 23] and, therefore are omitted here. We present scaled HDs for anisotropic growth models in the following sections.

model	g_1	g_2	g_3	g_4	R	S	K
RSOS (m=1)	-0.4795(5)	0.2553(1)	0.03772(5)	0.01075(5)	1.110(2)	0.292(1)	0.1649(5)
RSOS (m=2)	-0.709(1)	0.5547(4)	0.1216(2)	0.0512(1)	1.103(4)	0.294(2)	0.1664(5)
BD	-1.294(1)	1.8095(6)	0.712(1)	0.536(3)	1.080(2)	0.292(2)	0.163(1)
Eden	-1.770(2)	≈ 0.81	0.161(2)	0.0607(5)	0.26	0.22	0.093

Table 2. Asymptotic values for g_n for isotropic growth models. Dimensionless cumulant ratios $R = g_2/g_1^2$, $S = g_3/g_2^{3/2}$ and $K = g_4/g_2^2$ are also included. Lack of uncertainty in g_2 did not allow to determine errors in dimensionless ratios for Eden model.

4. Anisotropic radial growth

Radial interfaces were simulated using Eden models on square lattices. Two definitions are necessary to describe the models. Periphery sites are occupied sites with at least one empty nearest neighbor (NN), while growth sites are empty sites with at least one occupied NN. We investigated two versions. In the version Eden A, a growth site is chosen at random and occupied. For each choice, the time is increased by $1/N_g$, where N_g is the number of growth sites. In the version Eden D on a lattice, a periphery site and one of its NNs are randomly chosen. If the selected NN is empty, it receives a new particle, otherwise, the simulation runs to the next step. For each attempt, the time is incremented by $1/N_p$, where N_p is the number of periphery sites. In all versions, the growth starts with a single occupied site at the center of the lattice.

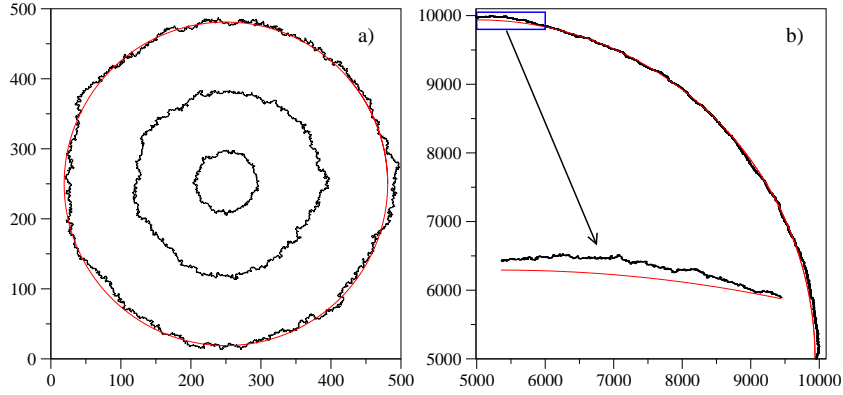


Figure 8. (a) Successive borders of on-lattice Eden D model for small sizes. The circle has the mean radius of the border. (b) First quarter of the border of a large on-lattice Eden D cluster. The inset shows a zoom of the cluster top.

The start point to check the equivalence between isotropic and anisotropic growth is the interface width w , defined as the standard deviation of the radius measured in relation to the center of the lattice. At early times, it behaves as a power law $w \sim t^\beta$, where the exponent $\beta = 1/3$ is expected for the KPZ class in $d = 1 + 1$. However, it is well known that Eden clusters are affected by the lattice-imposed anisotropy [27, 30]. More precisely, the axial direction $\langle 10 \rangle^\ddagger$ grows slightly faster than the diagonal $\langle 11 \rangle$ implying that the dispersion around the mean radius of the border is asymptotically ruled by a diamond-like shape that produces a crossover from $w \sim t^{1/3}$ at short times to $w \sim t$ at long times [27, 30]. Figure 8 shows the evolution of an Eden cluster where the faster growth along $\langle 10 \rangle$ can be seen.

Since different directions have different growth velocities, we must focus on the fluctuations along a fixed direction. The radius on a given direction is defined as the distance from the origin of the farthest cluster particle along that direction. Equation (2) can be applied for an arbitrary direction with the parameters v_∞ and Γ depending on it. At a time t , we have a collection of radii $\{r_1(t), r_2(t), \dots, r_N(t)\}$ along a given direction that are obtained from an ensemble of N independent simulations. We analyzed axial and diagonal directions that correspond to fastest and slowest growth directions in square lattices, respectively. We simulated 2.5×10^6 clusters of diameter 5×10^3 . Therefore, our statistics is performed with 10^7 points for each analyzed direction. A typical simulation of Eden A and D takes about 3 s and 7 s, respectively, in a CPU Intel Xeon 3.20GHz.

Figure 9 shows the interface width against time for Eden A computed for distinct directions. Interface width along direction $\langle 10 \rangle$ is larger than along direction $\langle 11 \rangle$ ($\Delta w \approx 6\%$), but they follow scaling laws in time with growth exponents $\beta_{10} = 0.335(7)$ and $\beta_{11} = 0.332(7)$, respectively, in excellent agreement with the KPZ universality class. Growth exponents for Eden A and D are shown in table 4. The inset B of figure 9

\ddagger Here, we have borrowed the crystallographic notation where $\langle 10 \rangle$ represents the directions of unitary vectors $\hat{x}, -\hat{x}, \hat{y}$ and $-\hat{y}$.

shows the analysis to determine the interface velocity for Eden A simulations. A higher velocity along axial direction is evident.

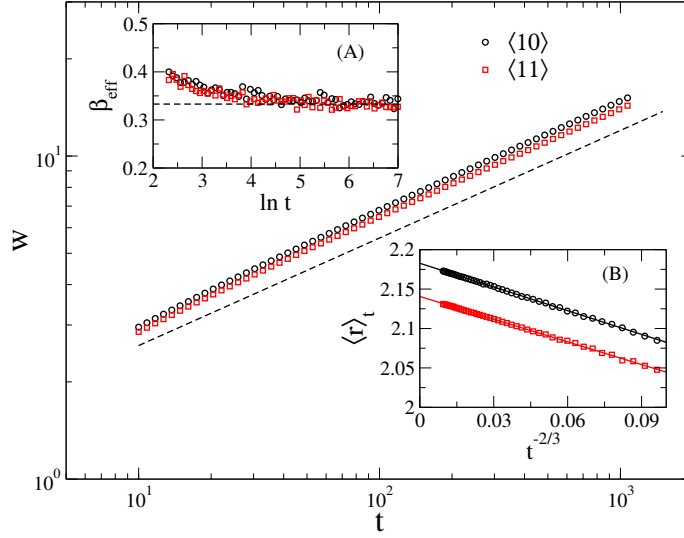


Figure 9. Interface width against time for on-lattice Eden A model. Inset (A): Effective growth exponent obtained from the derivative of $\ln w$ versus $\ln t$. Inset (B): Interface velocity against $t^{-2/3}$.

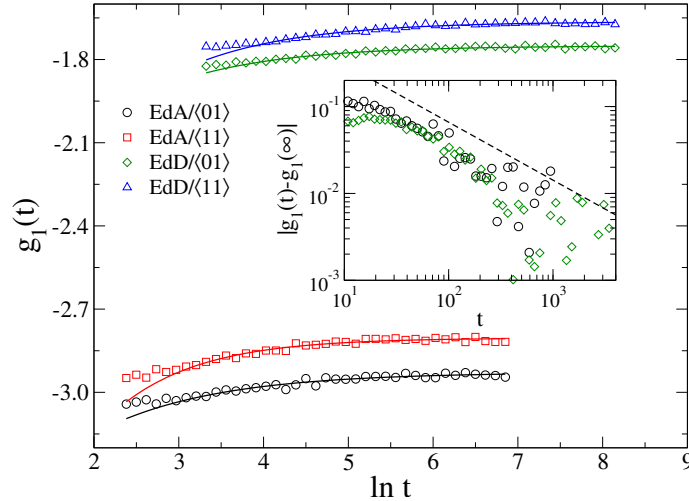


Figure 10. Determination of the non-universal quantity $\Gamma^{1/3}\langle\chi\rangle$ for on-lattice Eden models. Solid lines are non-linear regressions used to extrapolate g_1 . The inset shows the correction in g_1 against time. Dashed line has slope $-2/3$.

Corrections in g_1 are shown in figure 10 and are consistent with a decay $t^{-2/3}$ implying that the next leading term in equation (5) is $t^{-1/3}$, in analogy with isotropic growth models described in sections 2 and 3. The scaled cumulants against time for Eden A are shown in figure 11. Along the direction $\langle 10 \rangle$, we have found corrections consistent with t^{-1} for the second cumulant in both Eden A and D. Along direction

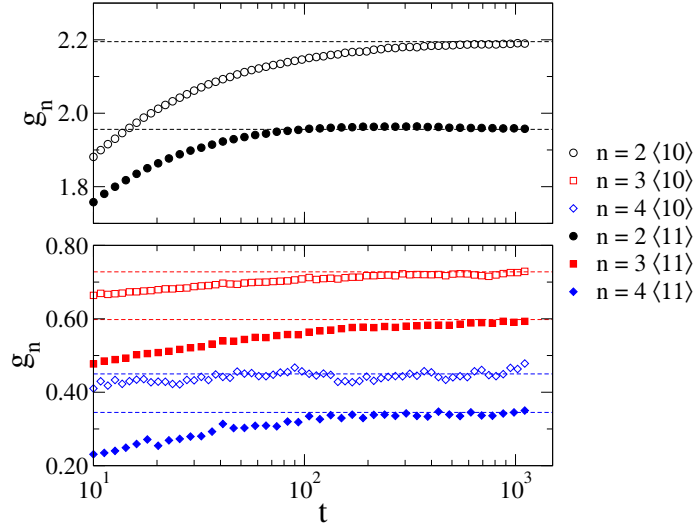


Figure 11. Scaled cumulants $g_n = \langle r^n \rangle_c / (s_\chi^n t^{n/3})$ for on-lattice Eden A along directions $\langle 10 \rangle$ (open symbols) and $\langle 11 \rangle$ (filled symbols). Lines are extrapolations to $t \rightarrow \infty$.

$\langle 11 \rangle$, g_2 has a slight non-monotonicity that complicates the extrapolation to long times. It was not possible to accurately resolve the scaling of the corrections for higher order cumulants but we have found that they do not decay slower than $t^{-2/3}$, as illustrated in figure 12. This indicates that the correction η is statistically independent of χ since, otherwise, corrections would decay as $t^{-1/3}$ for all higher order cumulants [8, 24]. Notice that many features of our current on-lattice simulations were observed for the isotropic growth, where $t^{-2/3}$ was found for g_1 , g_3 and g_4 and a non-monotonicity for g_2 . Table 3 shows the non-universal parameters v_∞ and g_n obtained in our simulations.

One can see in table 3 that the estimates for Γ_1 are slightly larger than those for Γ_2 for all investigated models, as observed for ballistic deposition on flat substrates. In the rest of this work, we use g_1 to investigate the scaling and universality of the models since it yields reliable estimates of Γ as discussed in section 2.

Table 3. Non-universal parameters of several anisotropic growth models with curved geometry. Acronyms: EdA (Eden A); EdD (Eden D); WRSOS (wedge restricted solid-on-solid); BDD (ballistic deposition droplet). The missing uncertainties for EdA/ $\langle 11 \rangle$ and EdD/ $\langle 11 \rangle$ are due to the lack of monotonicity in the second cumulant of the respective data.

Model	v_∞	g_1	g_2	g_3	g_4	Γ_1	Γ_2	$\langle \eta \rangle$
EdA/ $\langle 10 \rangle$	2.1824(3)	-2.927(3)	2.195(1)	0.728(2)	0.45(1)	4.51(1)	4.43(1)	1.7(1)
EdA/ $\langle 11 \rangle$	2.1401(3)	-2.807(3)	1.956	0.598(2)	0.345(9)	3.98(1)	3.73	1.1(1)
EdD/ $\langle 10 \rangle$	0.6186(1)	-1.754(2)	0.779(1)	0.1535(8)	0.057(1)	0.971(3)	0.938(2)	1.6(1)
EdD/ $\langle 11 \rangle$	0.61055(5)	-1.677(2)	0.695	0.127(2)	0.044(1)	0.85(1)	0.79	1.1(1)
WRSOS	0.41903(1)	-1.115(4)	0.3233(2)	0.0418(2)	0.0095(4)	0.250(3)	0.251(1)	-0.7(1)
BDD	2.13984(2)	-3.023(9)	2.331(5)	0.802(6)	0.49(1)	4.97(4)	4.85(2)	1.3(1)

Due to the anisotropy, the surfaces studied here do not have translational invariance

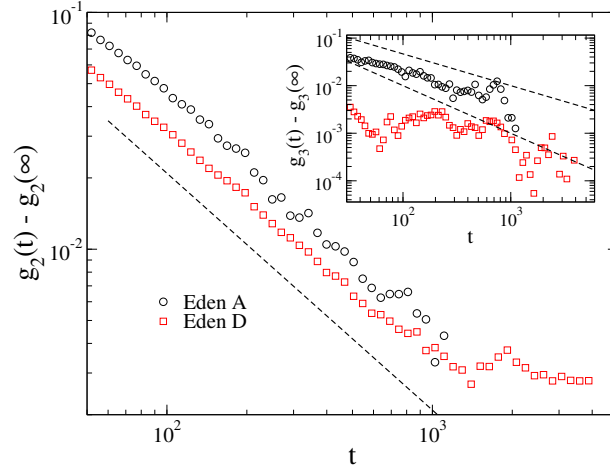


Figure 12. Corrections in the second cumulant for on-lattice Eden models along direction $\langle 10 \rangle$. The dashed line is a power law with exponent -1 . Inset: Corrections in the third cumulant. The dashed lines are power laws with exponents $-2/3$ and -1 .

Table 4. Universal quantities for several anisotropic models with curved geometry. Acronyms as in table 3. Cumulant ratios are defined by equations (12)-(14). The KPZ cumulant ratios correspond to the GUE distribution were taken from reference [17]. The lack of uncertainties in the second cumulant of EdA/ $\langle 11 \rangle$ and EdD/ $\langle 11 \rangle$ models did not allow the error propagation in the cumulant ratios.

Model	β	R	S	K
EdA/ $\langle 10 \rangle$	0.335(7)	0.256(1)	0.224(1)	0.093(2)
EdA/ $\langle 11 \rangle$	0.332(7)	0.25	0.22	0.090
EdD/ $\langle 10 \rangle$	0.338(9)	0.253(1)	0.223(2)	0.094(2)
EdD/ $\langle 11 \rangle$	0.334(10)	0.25	0.22	0.091
WRSOS	0.3326(5)	0.260(2)	0.227(1)	0.091(4)
BDD	0.335(2)	0.255(2)	0.225(2)	0.090(2)
KPZ	1/3	0.2592	0.2241	0.09345

so that two-point local quantities are not well-defined. So, the parameter A and, consequently, Γ cannot be directly measured if cumulants of χ are unknown. We then define

$$q' = \frac{r - v_\infty t}{s_\lambda g_1 t^{1/3}} \equiv \frac{q}{\langle \chi \rangle} \quad (15)$$

such that $\langle q' \rangle \rightarrow 1$ for $t \rightarrow \infty$. We can obtain the scaling law ruling the shift in the mean by plotting $1 - \langle q' \rangle$ against time using only the directly measurable parameters v_∞ and g_1 . The results are shown in figure 13, where we have obtained corrections very consistent with $t^{-1/3}$, already observed in many other systems [6, 7, 10, 21, 24]. The slow crossover to $t^{-1/3}$ observed for off-lattice simulations of Eden D [23] was not observed in the on-lattice version. Moreover, our previous off-lattice simulations of Eden B§ also

§ The difference between Eden D and Eden B is that any nearest neighbor can be chosen in the former

yielded a shift consistent with $t^{-1/3}$ [21]. An important remark is that the directly measurable amplitude in the scaling $1 - \langle q' \rangle \simeq -\langle \eta \rangle / g_1 t^{-1/3}$ also yields a way to obtain an estimate of $\langle \eta \rangle$. The values of $\langle \eta \rangle$ for anisotropic growth models are shown in table 3.

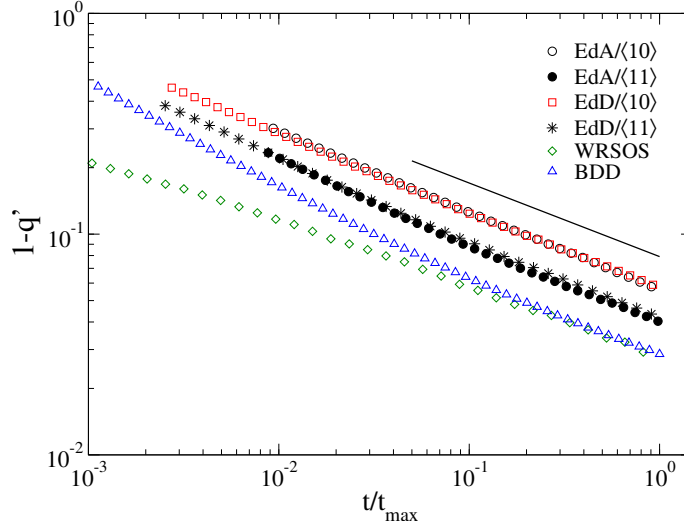


Figure 13. Scaling law describing the shift in the mean of scaled surface distributions for several anisotropic KPZ models. The time was scaled by the maximum growth time reached in each model in order to improve visibility. The solid line represents the power law $t^{-1/3}$. Acronyms as in table 3.

The dimensionless cumulant ratios obtained for both Eden models are shown in table 4. All ratios are universal and show remarkable agreement with GUE [17]. Notice that using dimensionless cumulant ratios as equations (12), (13) and (14), all cumulants of χ can be determined as functions of a single cumulant. This reasoning also applies to systems where χ is unknown raising a general strategy to probe universality in the interface distributions.

The top panel of figure 14 compares the distributions scaled accordingly equation (15) for Eden A in the direction $\langle 10 \rangle$ at different times with the GUE distribution scaled to a mean 1, $\chi^* = \chi / \langle \chi \rangle$. The distributions converge to GUE as time evolves but the shift is evident. If the estimated shift $\langle \eta \rangle = 1.7(1)$ is explicitly included by defining

$$q^* = \frac{r - v_\infty t - \langle \eta \rangle}{s_\lambda g_1 t^{1/3}}, \quad (16)$$

an excellent collapse between $P(q^*)$ and $P(\chi^*)$ is found (figure 14, bottom). Similar results are found for all investigated models.

5. Droplet growth

A paradigmatic model for the growth of a droplet morphology is the PNG model, where islands are randomly nucleated over existing ones and, once nucleated, they starts while only the empty ones are eligible in the latter.

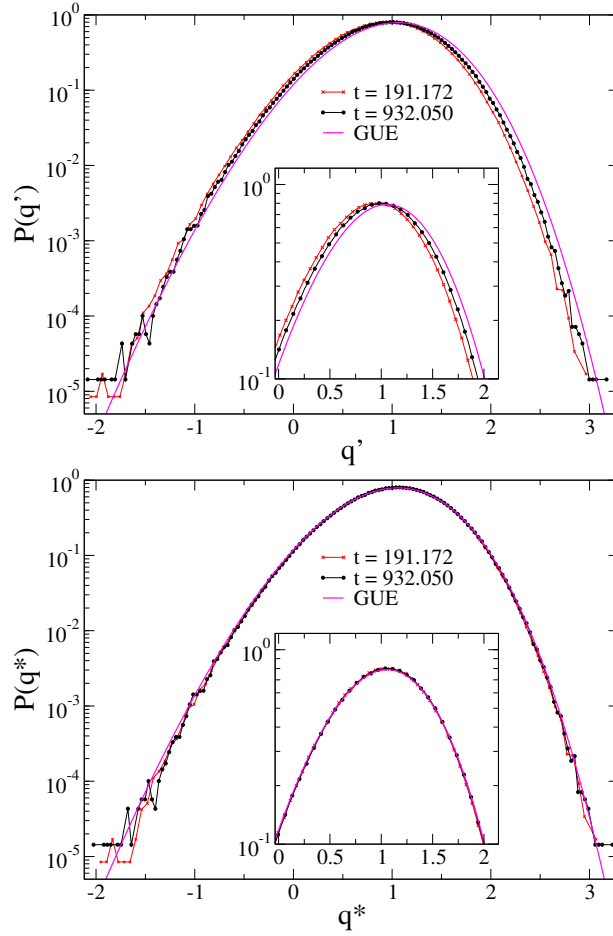


Figure 14. Top: Distributions for Eden A model at different growth times scaled accordingly equation (15). Bottom: Same distributions rescaled using equation (16) that includes the shift $\langle \eta \rangle = 1.7$. Insets show zooms around the peaks.

to grow laterally with constant velocity. If we consider an initial island that grows indefinitely from a single nucleation at the origin, the PNG model has an exact solution with an asymptotic droplet shape, where the height at the origin is given by equation (2) with exactly known parameters [17, 18]. PNG model simulations with droplet geometry have been performed to be compared with analytic results [17, 20]. Other models and initial conditions leading to droplet interfaces has also been investigated [10, 16, 36].

Droplet geometry can also be investigated in more complex models for which analytical results are currently unavailable as, for example, the BD model [38]. The growth of ballistic deposition droplets (BDD) starts with a single particle stuck to the origin of the system. Particles move ballistically along direction $-y$. If a particle visits a NN site of the aggregate, it irreversibly attaches to this position and becomes part of the aggregate. Let $L(t)$ be the size of the active region defined as the set of x -coordinates where a growth can be tried. When a particle deposition is tried the time is increased by $\Delta t = 1/L(t)$. Figure 15 shows a typical ballistic deposition droplet.

The non-universal parameters depend on the position in the active region in analogy

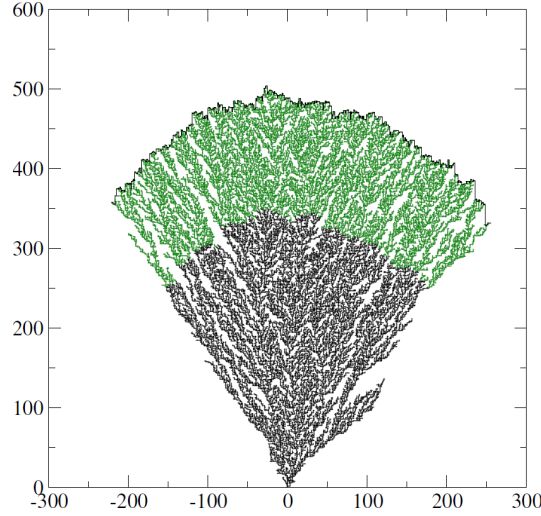


Figure 15. Ballistic deposition droplet after deposition of 56000 particles. The first half of deposited particles are depicted in black and the rest in green. The solid line represents the interface.

to the direction dependence in the Eden growth, whereas cumulant ratios and growth exponent do not. An ensemble of 2×10^7 samples were used to compute statistics. Growth of a cluster for $t = 10^4$ ($\sim 10^8$ particles) takes about 2 s in an CPU Intel Xeon 3.20GHz. For sake of conciseness, we present only results for the height fluctuations at the origin.

The curves used to determine the non-universal parameters (table 3) are shown in figures 16 and 17. The interface velocity has the usual behavior illustrated in the inset (B) of figure 16. However, BDD growth has strong corrections in the growth exponent as confirmed in the inset (A) of figure 16, in analogy to its flat counterpart [22, 37]. The correction in the mean converges in the usual way ($t^{-1/3}$) as shown in figure 13. Notice the presence of a crossover from a faster initial decay to the regime $t^{-1/3}$, analogous to the crossover observed for the isotropic Eden D model. Similarly to the isotropic growth, corrections in g_2 decays with an unusually small exponent $t^{-0.45}$.

The quantity g_1 approaches its asymptotic value as $t^{-2/3}$, the same correction obtained for flat simulations of isotropic growth models and for all versions of the anisotropic Eden model presented in section 4. The cumulant of order $n = 3$ also has a correction very close to $t^{-2/3}$ while the cumulant of order $n = 4$ has a correction faster than $t^{-2/3}$. In particular, the correction observed in g_3 is slower than that observed for the flat case, demonstrating the intricate and non-universal scenarios involving corrections in cumulants of order $n \geq 2$. The inset of figure 17 shows the corresponding corrections in g_n . Finally, the growth exponent and dimensionless cumulant ratios converge to the values expected for the KPZ class as shown in table 4.

We also considered the restricted solid-on-solid (RSOS) growth model [34] with a wedge initial condition, the WRSOS model. At each time step, a site in the growing zone (of size $L(t)$) is randomly selected and its height increased by 1 if the constraint

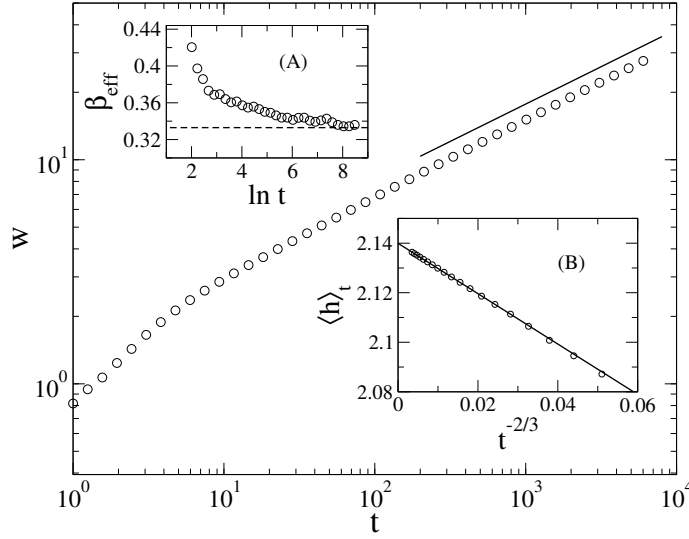


Figure 16. Interface width against time determined at center of the BDD model. Solid line is a power law $t^{1/3}$. The inset (A) shows the effective growth exponent while inset (B) shows the interface velocity against $t^{-2/3}$.

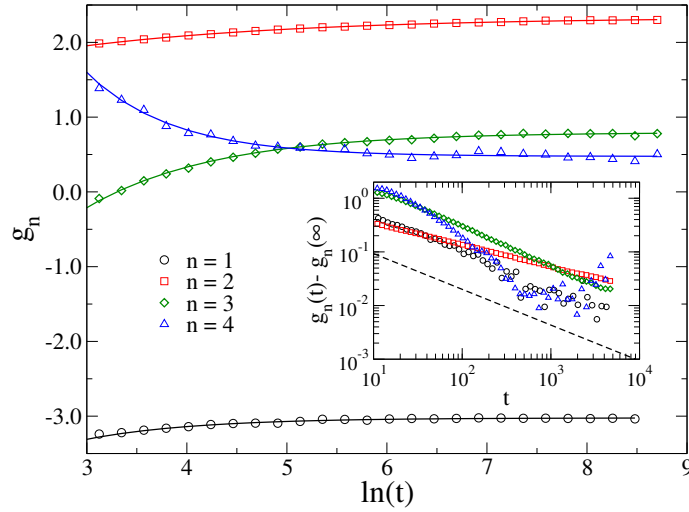


Figure 17. Semi-logarithmic plot of g_n against time for BDD model. Solid lines are the non-linear regression used to extrapolate the asymptotic values. Inset: Difference $g_n(t) - g_n(\infty)$ against time. The dashed line represents a decay $t^{-2/3}$.

$\Delta h = |h(j, t) - h(j \pm 1, t)| \leq 1$ is satisfied, otherwise, the deposition attempt is refused. The time is incremented by $\Delta t = 1/L(t)$ for each attempt. A wedge initial condition $h(j, 0) = |j|$, obeying $\Delta h = 1$, was considered. This initial condition implies a droplet growth since the radius of the growing zone increases with average velocity 1 while in the center ($\langle 10 \rangle$ direction) it has a smaller velocity (Tab. 3). Typical interfaces for distinct times are shown in figure 18.

Figure 19 shows the interface velocity at $j = 0, 10$, and 1000 . They converge to the same asymptotic value observed for the flat geometry [22], but differ at short times. This means a macroscopic shape, around the droplet center, that asymptotically moves

with constant velocity. The macroscopic shape is almost perfectly fitted by a parabola $h(x) = h(0) + ax^2$ in analogy with the solution of the KPZ equation with a sharp wedge initial condition [10]. The surface statistics was computed using an ensemble of 5×10^7 samples. A typical run up to $t = 3000$ takes about 0.2 s in a CPU Intel Xeon 3.20GHz. Non-universal parameters obtained for the height fluctuations at the origin $j = 0$ are shown in table 3. The universal quantities shown in table 4 exhibit an excellent agreement with the KPZ universality class. Far from the origin, the accordance with the KPZ conjecture is also observed, but the farther from the origin the slower the convergence.

The finite time corrections in the cumulants of WRSOS were also studied. The shift in the mean approaches the GUE one with the correction $t^{-1/3}$ (figure 13). The quantities g_n , for $n = 1, 3$ and 4 have a correction $t^{-2/3}$ while g_2 decays slightly faster than $t^{-2/3}$ (figure 20) in analogy with the flat simulations of RSOS.

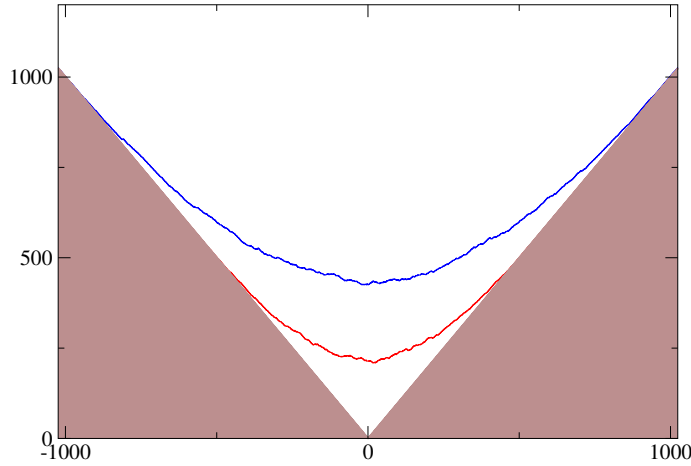


Figure 18. Typical interfaces obtained for WRSOS model for deposition times $t = 512$ (lower) and $t = 1024$ (upper) .

6. Discussion

We have performed extensive simulations of models belonging to KPZ universality class in order to probe the generality of the KPZ ansatz given by equation (5) and to determine further corrections to this equation. We have analyzed two classes of models. In the isotropic growth models, all points of the interface are used to perform averages, in contrast with the anisotropic ones, where the non-universal parameters vary along the surface, limiting the statistics to a single or a few points per sample.

During the numerical investigation, we have found out that the determination of the non-universal parameter Γ using the asymptotic value of the scaled second cumulant $g_2 = \langle h^2 \rangle_c / t^{2/3}$, has limited accuracy for some models, probably due to unknown puzzling corrections. We propose an alternative way to obtain Γ using the first cumulant derivative by means of the quantity $g_1 = s_\lambda 3(\langle h \rangle_t - v_\infty) t^{2/3} \rightarrow \Gamma^{1/3} \langle \chi \rangle$. It is important

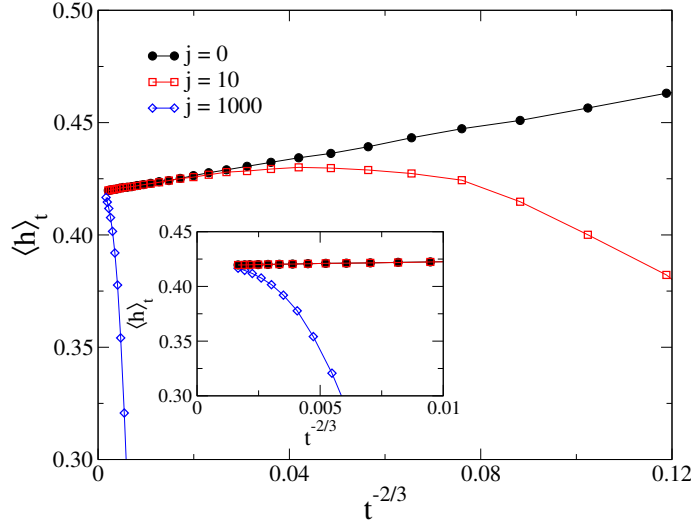


Figure 19. Interface velocity at different positions of WRSOS growth. The inset shows a zoom to show the asymptotic velocity.

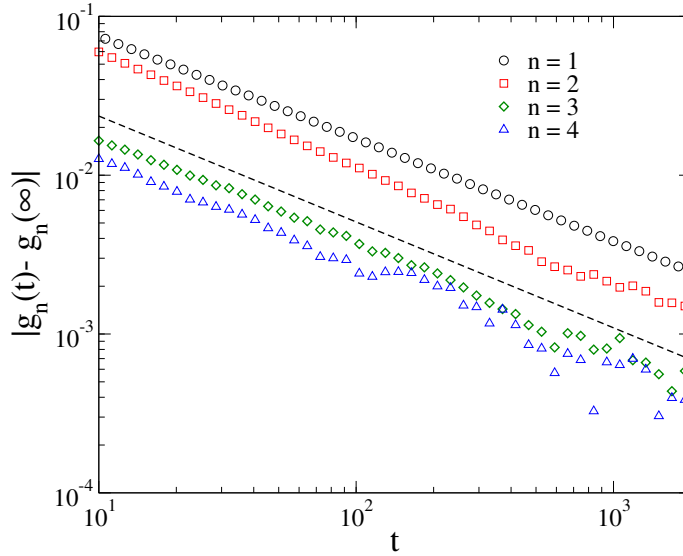


Figure 20. Time corrections in g_n for WRSOS model at $j = 0$. Dashed line is a power law $t^{-2/3}$.

to mention that noise in the numerical derivative counts against this method. Here, we used very large statistics to smooth the curves. However, if statistics is limited, as possibly in case of experiments, one can use numerical procedures to smooth the derivative [39]. We also considered scaled cumulants $g_n = \langle h^n \rangle_c / (s_\lambda^n t^{n/3})$ of third and fourth order to determine Γ . In general, we have observed that small deviations in Γ may suggest a fake violation of the generalized KPZ ansatz. Estimates taken from g_1 give very good agreement with the generalized KPZ conjecture for all investigated models. Dimensionless universal cumulant ratios are not very sensitive to corrections in cumulants as one can see in tables 2 and 4, in which excellent agreements with GOE and GUE distributions were found for flat and curved models, respectively.

Comparing the non-universal parameters for the flat RSOS and BD models (table 1) with its droplet counterparts (table 3), we see that within the error bars they are equal, with exception of the shift $\langle\eta\rangle$. On the other hand, in the Eden D model all parameters are different for on- and off-lattice simulations. This is expected, since the lattice constraint should change the strength of fluctuations.

In off-lattice simulations of the Eden model, we determined the shift in the height distributions and our results support a crossover from $t^{-2/3}$ to $t^{-1/3}$ in the first cumulant in relation to GUE, clarifying an apparent exception to the correction $t^{-1/3}$ recently suggested by Takeuchi [23].

From the numerical side, we propose that the estimates of Γ taken from the first cumulant are more reliable than those obtained from the higher order ones since, in principle, we know little about the nature of terms beyond χ in the KPZ ansatz. These terms may introduce anomalous corrections as, for example, in the case of statistical dependence among random variables. In fact, in contrast to the $t^{-2/3}$ correction in g_1 obtained for all investigated models, we have not found a standard in the corrections of higher order cumulants.

The first cumulant derivative analysis for all investigated models are in agreement with an additional term generalizing the KPZ ansatz to

$$h = v_\infty t + s_\lambda (\Gamma t)^{1/3} \chi + \eta + \zeta t^{-1/3} + \dots \quad (17)$$

Corrections $\mathcal{O}(t^{-1/3})$ were reported also for some analytically tractable KPZ class models [24], suggesting that this term is a general property of KPZ systems. However, in the solution of the KPZ equation with a sharp wedge initial condition a correction $\mathcal{O}(t^{-1})$ was reported [10, 12], possibly because the non-universal parameter ζ is zero in this case.

The generalized ansatz given by equation (17) allows one to speculate on the origin of the complicated corrections obtained for cumulants of order $n \geq 2$. For instance, if η and ζ are statistically dependent, a term t^{-1} will appear in the second cumulant in addition to the leading term $t^{-2/3}$. If the amplitude of t^{-1} is much larger than that of $t^{-2/3}$, one would observe a slow crossover to $t^{-2/3}$. This scenario explains the correction in g_2 observed for the RSOS model. Still in the field of speculation, if χ and η are statistically dependent the two first leading terms of the correction in g_2 would be $t^{-1/3}$ and $t^{-2/3}$. Again, if the amplitude of the second leading term is much larger than the first one, we could explain the anomalously slow exponent observed in the BD model as a crossover between the two scaling regimes. However, a double power law regression, as that used to determine the shift correction in Eden D, does not support such a conjecture.

Now we discuss how universality in height distributions can be checked without an explicit knowledge of the parameter Γ . Performing a rescaling using the directly measurable parameters from tables 1, 2 and 3, we can define the variable:

$$q^* = \frac{h - v_\infty t - \langle\eta\rangle}{s_\lambda g_1 t^{1/3}}.$$

The scaled distributions for this variable, $P(q^*)$, agree remarkably well with the GUE distribution scaled to unitary mean, $\chi^* = \chi/\langle\chi\rangle$, as shown in Figure 21, top. Our entire analysis depends implicitly on $\langle\chi\rangle$, but does not provide information about it beyond its negative sign, since $g_1 < 0$. We can alternatively define a variable with asymptotic unitary variance using the measurable parameter g_2 rather than g_1 to find

$$q^{**} = \frac{h - v_\infty t - \langle\eta\rangle}{g_2^{1/2} t^{1/3}} \rightarrow \frac{\chi}{\sqrt{\langle\chi^2\rangle_c}}, \quad (18)$$

with a mean depending on the dimensionless and measurable cumulant ratio R given by

$$\langle q^{**} \rangle \equiv \text{sgn}(\langle\chi\rangle) R^{-1/2} = -1.9641.$$

Again, as shown in figure 21 bottom, a very good collapse is obtained. We remark that this kind of analysis could be very profitable to probe the universality of height distributions in systems where they are not known *a priori*, such as high dimensional KPZ growth or models in universality classes without analytical counterparts. Although

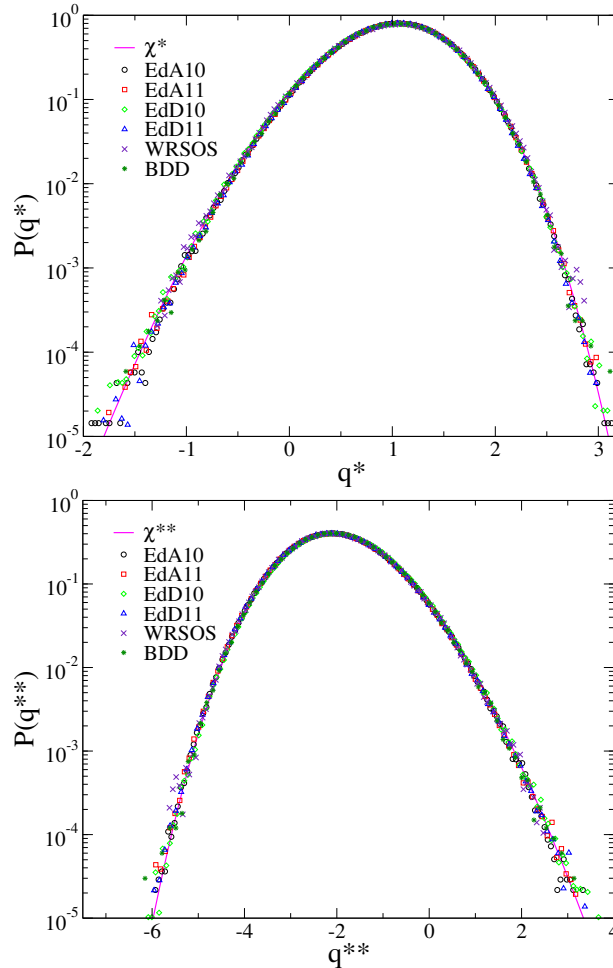


Figure 21. Rescaled distributions for all investigated models using equations (16) and (18) in top and bottom panels, respectively.

the complete distribution are not determined, once $\langle\chi\rangle$ or $\sqrt{\langle\chi^2\rangle_c}$ is fixed everything concerning the HD can be derived.

Acknowledgments

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